Circle actions on 7-manifolds

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Abstract

We determine the homeomorphism types of those 2-connected 7-manifolds that admit regular circle actions.

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1 Introduction

In this paper all manifolds under consideration are closed, oriented and topological, unless otherwise stated. Moreover, all homeomorphisms and diffeomorphisms considered are to be orientation preserving. Given a positive integer n let S^n be the diffeomorphism type of the unit $n-$sphere in $(n+1)-$dimensional Euclidean space $\mathbb{R}^{n+1}$.

Definition 1.1. A circle action $S^1 \times M \to M$ on a manifold M is called *regular* if this action is free and the orbit space $N := M/S^1$ (with quotient topology) is a manifold.□

For a given manifold M one can ask
Problem 1.2. does M admit a regular circle action?□

Solutions to Problem 1.2 can have direct implications in contact topology. For example, the Boothby-Wang theorem implies that, in the smooth category, the existence of a regular circle action on a manifold M is a necessary condition to the existence of a regular contact form on M (see [7, p.341] for instance).

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Problem 1.2 has been solved for all $1-$connected $5-$manifolds by Duan and Liang in [5]. In particular, it was shown that all $1-$connected $4-$manifolds with second Betti number r can be realized as the orbit spaces of some regular circle actions on the single $5-$manifold $\#_{r-1}S^2 \times S^3$, the connected sums of $r-1$ copies of the product manifold $S^2 \times S^3$. In this paper we study Problem 1.2 for the $2-$connected $7-$manifolds.

Based on the construction of S^3-bundles over S^4 from the structure group $\pi_3(SO(4)) = \mathbb{Z} \oplus \mathbb{Z}$ and resorting to the surgery theory for the $2-$connected $7-$manifolds we shall present explicitly a family $M^c_{l,k}$ of $2-$connected $7-$manifolds with $c \in \{0,1\}$ and $l,k \in \mathbb{Z}$ in Section 3.2. In terms of the manifolds $M^c_{l,k}$ our main result is stated below, where $\mathbb{N}$ denotes the set of all non–negative integers.

Theorem 1.3. Let M be a $2-$connected $7-$manifold. Then M admits a regular circle action if and only if M is homeomorphic to

$$\#_{2r}S^3 \times S^4 \# M^c_{6m,(1+c)k}$$

where $c \in \{0,1\}$, $r \in \mathbb{N}$ and $m,k \in \mathbb{Z}$.$\square$

In the course to establish Theorem 1.3 we obtain also a classification on the $6-$manifolds that can appear as the orbit manifolds of some regular circle actions on $2-$connected $7-$manifolds, see Lemma 2.1 and Lemma 2.2 in Section 2. In addition, our method applies equally to obtain a result analogue to Theorem 1.3 in the smooth category, see Theorem 4.1 in Section 4.

2 The homeomorphism types of the orbit spaces

In this section we determine the homeomorphism types of those $6-$manifolds which can appear as the orbit spaces of some regular circle actions on $2-$connected $7-$manifolds. Note that a manifold M admits a regular circle action if and only if it is the total space of a principal S^1-bundle $M \to N$ with base space N a manifold.

Let M be a $2-$connected $7-$manifold that admits a regular circle action with orbit space $N := M/S^1$. From the homotopy exact sequence

$$0 \to \pi_2(M) \to \pi_2(N) \to \pi_1(S^1) \to \pi_1(M) \to \pi_1(N) \to 0$$

of the S^1-fibration $M \to N$ one finds that

$$\pi_1(N) = 0;\ \pi_2(N) \cong \pi_1(S^1) \cong \mathbb{Z}.$$

Consequently, N is a $1-$connected $6-$manifold with $H_2(N) \cong \mathbb{Z}$.

Conversely, for each $1-$connected $6-$manifold N with $H_2(N) \cong \mathbb{Z}$ let $t \in H^2(N) = \mathbb{Z}$ be a generator and let

$$S^1 \hookrightarrow N_t \to N$$

be the oriented circle bundle over N with Euler class t. From the homotopy exact sequence of this fibration we find that the total space N_t is $2-$connected. Summarizing we get

Lemma 2.1. Let $S^1 \times M \to M$ be a regular circle action on a $2-$connected $7-$manifold M with orbit space N. Then N is a $1-$connected $6-$manifold with $H_2(N) \cong \mathbb{Z}$.

Conversely, every $1-$connected $6-$manifold N with $H_2(N) \cong \mathbb{Z}$ can be realized as the orbit space of some regular circle action on a $2-$connected $7-$manifold.$\square$

In view of Lemma 2.1 the classification of those $1-$connected $6-$manifolds N with $H_2(N) \cong \mathbb{Z}$ amounts to a crucial step toward a solution to Problem 1.2. In terms of the known invariants for $1-$connected $6-$manifolds due to Jupp [10] and Wall [22], we can enumerate all these manifolds in the next result. For convenience denote by Θ the set of all $1-$connected $6-$manifolds N whose integral cohomology satisfies

$$H^r(N) = \begin{cases} \mathbb{Z} \text{ if } r = 0, 2, 4, 6; \\ 0 \text{ otherwise.} \end{cases}$$

For each manifold $N \in \Theta$ fix a generator t for $H^2(N)$ and a generator x for $H^4(N)$ respectively such that the evaluation $\langle t \cup x, [N] \rangle$ of the cup product $t \cup x$ on the fundamental class $[N]$ of N is 1. Consider functions

$$k, p : \Theta \to \mathbb{Z};\ \ \varepsilon : \Theta \to \{0,1\};\ \ \delta : \Theta \to \{0,1\}$$

defined by the following rule

i) $t^2 = k(N)x$;

ii) the second Stiefel-Whitney class $w_2(N)$ and the first Pontrjagin class $p_1(N)$ of N are given by $\varepsilon(N)t \mod 2$ and $p(N)x$, respectively;

iii) the class $\Delta(N) \equiv \delta(N)x \mod 2 \in H^4(N; \mathbb{Z}_2)$ is the Kirby-Siebenmann invariant of N

(The Kirby-Siebenmann invariant $\Delta(V)$ of a manifold V is the obstruction to lift the classifying map $V \to BTOP$ for the stable tangent bundle of V to BPL, where $BTOP$ and BPL are the classifying spaces for the stable TOP bundles and PL bundles, respectively).

[10, Theorem 1] implies that two manifolds $N_1, N_2 \in \Theta$ are homeomorphic if and only if either

$$(k(N_1), p(N_1), \varepsilon(N_1), \delta(N_1)) \;=\; (k(N_2), p(N_2), \varepsilon(N_2), \delta(N_2))$$
$$\text{or } (k(N_1), p(N_1), \varepsilon(N_1), \delta(N_1)) = (-k(N_2), -p(N_2), \varepsilon(N_2), \delta(N_2)).$$

This shows that $\{k, p, \varepsilon, \delta\}$ is a set of complete invariants for elements in Θ. Furthermore, we have

Lemma 2.2. For each $1-$connected $6-$manifold M with $H_2(M) \cong \mathbb{Z}$ there exists an integer $r \in \mathbb{N}$ and a manifold $N \in \Theta$ unique up to homeomorphism so that $M = \#_r S^3 \times S^3 \# N$.

Moreover, the system $\{k, p, \varepsilon, \delta\}$ is a set of complete invariants for elements N in Θ that is subject to the following constraints:

i) if $k(N)$ is odd, then $\varepsilon(N) = 0$ and

$$p(N) = 24m + 4k(N) + 24\delta(N);$$

ii) if $k(N)$ is even, then

$$p(N) = \begin{cases} 24m + 4k(N) + 24\delta(N) & \text{if } \varepsilon(N) = 0 \\ 48m + k(N) + 24\delta(N) & \text{if } \varepsilon(N) = 1 \end{cases},$$

for some $m \in \mathbb{Z}$.

In addition, a manifold $N \in \Theta$ has a smooth structure if and only if $\delta(N) = 0$.

Proof. This is a direct consequence of [10, Theorem 0; Theorem 1]. In particular, the expressions of the function p are deduced from the relation on $H^6(N)$ (see [10, Theorem 1]):

$$(2ct + \varepsilon(N)t)^3 \equiv (p(N)x + 24\delta(N)x)(2ct + \varepsilon(N)t) \bmod 48$$

for all $c \in \mathbb{Z}$.$\square$

3 Circle bundles over $N \in \Theta$

Lemma 2.2 singles out the family Θ of $1-$connected $6-$manifolds which plays a key role in presenting the orbit spaces of regular circle actions on $2-$connected $7-$manifolds. In this section we determine the homeomorphism type of the total space N_t of the circle bundle over $N \in \Theta$ with Euler class the fixed generator $t \in H^2(N) \cong \mathbb{Z}$. For this purpose we shall recall in Section 3.1 the definition of the known invariant system associated to $2-$connected $7-$manifolds; In Section 3.2 we give an explicit construction of the manifolds $M^c_{l,k}$ appearing in Theorem 1.3. The main result in this section is Lemma 3.3, which identify the homeomorphism types of the manifolds N_t with certain $M^c_{l,k}$.

3.1 Invariants for 2−connected 7−manifolds

Recall from Eells, Kuiper [6], Kreck, Stoltz [11] and Wilkens [23] that associated to each 2−connected 7−manifold M there is a system $\{H, q_1, b, \Delta, \mu, s_1\}$ of invariants characterized by the following properties:

i) H is the forth integral cohomology group $H^4(M)$;

ii) $q_1(M) \in H^4(M)$ is the first spin characteristic class of M(see [20]);

iii) $b : \tau(H) \otimes \tau(H) \to \mathbb{Q}/\mathbb{Z}$ is the linking form on the torsion part $\tau(H)$ of the group H;

iv) $\Delta(M) \in H^4(M; \mathbb{Z}_2)$ is the Kirby–Siebenmann invariant of M.

where the relation between the class $q_1(M)$ and the first Pontrjagin class $p_1(M)$ is $2 \cdot q_1(M) = p_1(M)$. Furthermore, if the manifold M is smooth (resp. topological) and bounds an 8−dimensional manifold W with the induced map $j^* : H^4(W, M; \mathbb{Q}) \to H^4(W; \mathbb{Q})$ an isomorphism, then

v) the invariant $\mu \in \mathbb{Q}/\mathbb{Z}$ (resp. $s_1 \in \mathbb{Q}/\mathbb{Z}$) is defined and its value is given by the formula below
$\mu(M) \equiv -\frac{1}{2^5 \cdot 7}\sigma(W) + \frac{1}{2^7 \cdot 7}p_1^2(W) - \frac{1}{2^6 \cdot 3}z^2 \cdot p_1(W) + \frac{1}{2^7 \cdot 3}z^4 \bmod \mathbb{Z}$
(resp. $s_1(M) = 28\mu(M)$)

where $z \in H^2(W)$ is a class whose mod 2 reduction gives the second Stiefel–Whitney class $w_2(W)$; $\sigma(W)$ is the signature of the bilinear form

$$H^4(W, M; \mathbb{Q}) \times H^4(W, M; \mathbb{Q}) \xrightarrow{\cup} H^8(W, M; \mathbb{Q}) = \mathbb{Q}$$

on $H^4(W, M; \mathbb{Q})$, and where $p_1^2(W)$, $z^2 \cdot p_1(W)$ and z^4 are the characteristic numbers

$$\langle p_1(W) \cup j^{*-1}p_1(W), [W, M]\rangle,\ \langle z^2 \cup j^{*-1}p_1(W), [W, M]\rangle,$$
$$\langle z^2 \cup j^{*-1}z^2, [W, M]\rangle,$$

respectively.

Example 3.1. Let N_t be the total space of the oriented circle bundle over $N \in \Theta$ with Euler class the fixed generator $t \in H^2(N) \cong \mathbb{Z}$. Then the system $\{H, q_1, b, \Delta, \mu, s_1\}$ of invariants for the manifold N_t can be expressed in terms of the invariants for the 6−manifold N introduced in Lemma 2.2 as follows. For simplicity we write p, k, ε and δ in the places of $p(N)$, $k(N)$,$\varepsilon(N)$ and $\delta(N)$, respectively.

i) $H^4(N_t) = \mathbb{Z}_k$ with generator $\pi^*(x)$, where $\pi : N_t \to N$ is the bundle projection and

$\mathbb{Z}_k = \begin{cases} \mathbb{Z} & \text{if } k = 0 \\ \mathbb{Z}/k\mathbb{Z} & \text{if } k \neq 0 \end{cases};$

ii) $\Delta(N_t) \equiv \frac{1+(-1)^k}{2} \cdot \delta \bmod 2$;

iii) $q_1(N_t) \equiv \frac{p+\varepsilon k}{2} \bmod k$;

iv) $b(\pi^*(x), \pi^*(x)) \equiv \frac{1}{k} \bmod \mathbb{Z}$;

v)$\mu(N_t) \equiv -\frac{|k|}{2^5 \cdot 7k} + \frac{(p+k)^2}{2^7 \cdot 7k} + \frac{(\varepsilon-1)(2p+k)}{2^7 \cdot 3} \bmod \mathbb{Z}$;

vi) $s_1(N_t) = 28\mu(N_t)$.

Firstly, from the section

$$H^2(N) \overset{\cup t}{\to} H^4(N) \overset{\pi^*}{\to} H^4(N_t) \to 0$$

in the Gysin sequence (see [16, p.143]) of the fibration $N_t \overset{\pi}{\to} N$ and from the relation $t^2 = kx$ on $H^4(N)$ we find that $H^4(N_t) = \mathbb{Z}_k$ with generator $\pi^*(x)$. This shows i).

Next, let $f : N \to BTOP$ be the classifying map for the stable tangent bundle of N, then, as is clear, the classifying map for the stable tangent bundle of N_t is given by the composition $f \circ \pi : N_t \to N \to BTOP$. It follows that the Kirby-Siebenmann invariant $\Delta(N_t)$ of the manifold N_t is $\pi^*\Delta(N) \equiv \delta\pi^*(x) \bmod 2$. This shows ii).

To calculate the remaining invariants q_1, b, μ, s_1 of the manifold N_t we make use of the disk bundle $W_t \overset{\pi_0}{\to} N$ associated with the oriented $2-$plane bundle ξ_t over N with Euler class t. If $\varepsilon = 1$, It follows from the decomposition $TW_t = \pi_0^*TN \oplus \pi_0^*\xi_t$ for the tangent bundle of W_t that $w_2(W_t) = 0$ and $q_1(W_t) = \frac{p+k}{2}\pi_0^*x$. So from the relation $\partial W_t = N_t$ we get

$$q_1(N_t) = \tfrac{p+k}{2}\pi^*x.$$

If $\varepsilon = 0$, from the decomposition $TN_t \cong \pi^*TN \oplus \varepsilon^1$ we get

$$q_1(N_t) = \tfrac{p}{2}\pi^*(x).$$

This shows iii).

To compute the linking form b of N_t we can assume that $k \neq 0$ in view of the presentation $H^4(N_t) = \mathbb{Z}_k$ with generator $\pi^*(x)$. Consider the commutative ladder of exact sequences

$$\begin{array}{ccccccccc} 0 & \to & H^4(W_t, N_t) & \overset{j^*}{\to} & H^4(W_t) & \overset{i^*}{\to} & H^4(N_t) & \to & 0 \\ & & \cong\uparrow \phi & & \uparrow \pi_0^* & & \uparrow id & & \\ 0 & \to & H^2(N) & \overset{\cup t}{\to} & H^4(N) & \overset{\pi^*}{\to} & H^4(N_t) & \to & 0 \end{array}$$

with ϕ the Thom isomorphism (see [16, p.106]). Since $y := \phi(t)$ is a generator of $H^4(W_t, N_t)$ with $j^*(y) = \pi_0^*(t^2) = k\pi_0^*(x)$ and since $\pi^*(x) = i^*\pi_0^*(x)$ we get

$$b(\pi^*(x), \pi^*(x)) \equiv \tfrac{1}{k} < y \cup \pi_0^* x, [W, M] > \equiv \tfrac{1}{k} \operatorname{mod} \mathbb{Z}.$$

This shows iv).

Since the induced map $j^* : H^4(W_t, N_t; \mathbb{Q}) \to H^4(W_t; \mathbb{Q})$ is clearly an isomorphism when $k \neq 0$, the invariants μ and s_1 are defined for N_t. Moreover, from the Poincaré duality and the relation $j^*(y) = k\pi_0^*(x)$ we get $\sigma(W_t) = \frac{|k|}{k}$. From the decomposition $TW_t \cong \pi_0^*(TN \oplus \xi_t)$ we get, in addition to

$$p_1(W_t) = \pi_0^*(p_1(N) + t^2) = (p + k)\pi_0^*(x),$$

that

$$w_2(W_t) \equiv (\varepsilon + 1)\pi_0^*(t) \operatorname{mod} 2.$$

Therefore we can take $z = (1 - \varepsilon)\pi_0^*(t)$ in the formulae for μ and s_1, and as a result

$$z^2 = (1 - \varepsilon)^2 \pi_0^*(t^2) = k(1 - \varepsilon)^2 \pi_0^*(x);$$

As the group $H^4(W_t, N_t) = \mathbb{Z}$ is generated by $y = \phi(t)$ with the relation $j^*(y) = k\pi_0^*(x)$, the isomorphism

$$H^4(W_t, N_t) \otimes H^4(W_t) \xrightarrow{\cup} H^8(W_t, N_t)$$

by the Poincaré duality, together with the formulae for $p_1(W_t)$ and z^2 above, implies the next relations

$$z^2 p_1(W_t) = (1 - \varepsilon)^2 (p + k);\ p_1^2(W_t) = \tfrac{1}{k}(p + k)^2;\ z^4 = k(1 - \varepsilon)^4.$$

Substituting these values in the formulae for μ and s_1 yields v) and vi), respectively.

This completes the computation of the invariant system $\{H, \Delta, q_1, b, s_1, \mu\}$ for the manifolds N_t.$\square$

3.2 The construction of the 7−manifolds $M_{l,k}^c$, $c \in \{0, 1\}$

Identify the algebra $\mathbb{H}$ of quaternions with the 4−dimensional Euclidean space $\mathbb{R}^4$ and regard the set of unit quaternions as the 3−dimensional sphere S^3. For a pair (l, k) of integers let $f_{l,k} : S^3 \to SO(4)$ be the map defined by

$$f_{l,k}(u)v = u^{l+k} v u^{-l},\ v \in \mathbb{R}^4,$$

where quaternion multiplication is understood on the right hand side of the formula. Let $M^0_{l,k}$ (resp. $W^0_{l,k}$) be the total space of the S^3-bundle (resp. D^4-bundle) over the sphere S^4 with characteristic map $[f_{l,k}] \in \pi_3(SO_4)$ and bundle projection

$$\pi_M : M^0_{l,k} \to S^4 \text{ (resp. } \pi_W : W^0_{l,k} \to S^4).$$

It is clear that $M^0_{l,k}$ is a $2-$connected $7-$manifold which carries a canonical smooth structure. What we shall also need is an analogue of the manifold $M^0_{l,k}$ in the non–smooth category when $k \equiv 0 \bmod 2$, called $M^1_{l,k}$, specified below.

For an m–manifold W with boundary write $\mathcal{S}^{TOP}(W)$ for the set of homeomorphism classes of the m–manifolds W', together with homotopy equivalences $h : (W', \partial W') \to (W, \partial W)$ (see [13, Chapter 2]). Let $[W', h]$ denote the equivalence class of (W', h). Furnish the two sets $\mathcal{S}^{TOP}(M^0_{l,k})$ and $\mathcal{S}^{TOP}(W^0_{l,k})$ with the structures of abelian groups so that the one to one correspondences

$$\mathcal{S}^{TOP}(M^0_{l,k}) \xrightarrow{\eta} [M^0_{l,k}, G/TOP] \text{ and } \mathcal{S}^{TOP}(W^0_{l,k}) \xrightarrow{\eta} [W^0_{l,k}, G/TOP]$$

in the surgery exact sequence are isomorphisms, where G (resp. TOP) is the monoid of self homotopy equivalences of S^{n-1} (resp. the group of origin-preserving homeomorphisms of $\mathbb{R}^n$) in the stable range (see [2, p.24-25] and [13, p.40-44]). On the other hand, as the groups $H^i(M^0_{l,k}; \pi_i(G/TOP))$ and $H^i(W^0_{l,k}; \pi_i(G/TOP))$ are trivial except for $i = 4$ (see [13, P.43-44]) , the obstruction theory implies that the primary obstructions to null-homotopy induce isomorphisms (see [4, p.179] and [9, p.192])

$$\begin{aligned} [M^0_{l,k}, G/TOP] &\xrightarrow{d} H^4(M^0_{l,k}; \pi_4(G/TOP)) = H^4(M^0_{l,k}); \\ [W^0_{l,k}, G/TOP] &\xrightarrow{d} H^4(W^0_{l,k}; \pi_4(G/TOP)) = H^4(W^0_{l,k}). \end{aligned}$$

Then consider the following commutative diagram:

$$\begin{array}{ccccc} \mathcal{S}^{TOP}(W^0_{l,k}) & \underset{\cong}{\xrightarrow{\eta}} & [W^0_{l,k}, G/TOP] & \underset{\cong}{\xrightarrow{d}} & H^4(W^0_{l,k}) \cong \mathbb{Z} \\ i^* \downarrow & & i^* \downarrow & & i^* \downarrow \\ \mathcal{S}^{TOP}(M^0_{l,k}) & \underset{\cong}{\xrightarrow{\eta}} & [M^0_{l,k}, G/TOP] & \underset{\cong}{\xrightarrow{d}} & H^4(M^0_{l,k}) \cong \mathbb{Z}_k \end{array}$$

where $i^* : \mathcal{S}^{TOP}(W^0_{l,k}) \to \mathcal{S}^{TOP}(M^0_{l,k})$ sends each $[W, h]$ to the restriction $[\partial W, h|_{\partial W}]$. Fix a generator ι of $H^4(S^4)$ as in [3]. Write $\left[W^1_{l,k}, h_W\right]$ for the generator $(d \circ \eta)^{-1}(\pi^*_W(\iota))$ of the cyclic group $\mathcal{S}^{TOP}(W^0_{l,k}) = \mathbb{Z}$ and set

$$\left(M^1_{l,k}, h_M\right) := \left(\partial W^1_{l,k}, h_W|_{\partial W^1_{l,k}}\right).$$

Clearly, the 7−manifold $M^1_{l,k}$ is 2−connected and unique up to homeomorphism with

$$d \circ \eta\left(\left[M^1_{l,k}, h_M\right]\right) = \pi_M^*(\iota).$$

Example 3.2. The invariant system $\{H, \Delta, q_1, b, s_1, \mu\}$ of the manifolds $M^c_{l,k}$, $c \in \{0,1\}$, has been computed by Crowley and Escher in [3] for the case of $c = 0$. We extend their calculation as to include the exceptional case of $c = 1$.

i) $H^4(M^c_{l,k}) = \mathbb{Z}_k$ with a generator

$\kappa = \begin{cases} \pi_M^*(\iota) & \text{if } c = 0 \\ (\pi_M \circ h_M)^*(\iota) & \text{if } c = 1 \end{cases};$

ii) $b(\kappa, \kappa) \equiv \frac{1}{k} \bmod \mathbb{Z}$;

iii) $\Delta(M^c_{l,k}) \equiv \frac{1+(-1)^k}{2} \cdot c \bmod 2$;

iv) $q_1(M^c_{l,k}) \equiv 2l + 12c \bmod k$;

v) $s_1(M^c_{l,k}) \equiv \frac{(2l+k+12c)^2-|k|}{8k} \bmod \mathbb{Z}$;

vi) $\mu(M^0_{l,k}) \equiv \frac{(k+2l)^2-|k|}{28 \cdot 8k} \bmod \mathbb{Z}$.

Firstly, since $h_M : M^1_{l,k} \to M^0_{l,k}$ is a homotopy equivalence, we get i) and ii) from the relations $H^4(M^0_{l,k}) = \mathbb{Z}_k$ with generator $\pi_M^*(\iota)$ and $b(\pi_M^*(\iota), \pi_M^*(\iota)) \equiv \frac{1}{k} \bmod \mathbb{Z}$ for $c = 0$.

Next, since the map $\mathcal{S}^{TOP}(M^0_{l,k}) \xrightarrow{\Delta} H^4(M^0_{l,k}; \mathbb{Z}_2)$ of taking Kirby– Siebenmann class is a surjective homomorphism (see [19, Theorem 15.1]), and since $\left[M^1_{l,k}, h_M\right]$ is a generator of the cyclic group $\mathcal{S}^{TOP}(M^0_{l,k})$, then we have

$$\Delta(M^1_{l,k}) = \Delta(\left[M^1_{l,k}, h_M\right]) = \tfrac{1+(-1)^k}{2}\kappa \bmod 2.$$

This shows iii).

In order to find the formula of $q_1(M^c_{l,k})$ we compute the first Pontrjagin class $p_1(W^c_{l,k})$ of $W^c_{l,k}$. Let α be a generator of $H^4(W^c_{l,k}) = \mathbb{Z}$ such that

$$\alpha = \begin{cases} \pi_W^*(\iota) & \text{if } c = 0; \\ (\pi_W \circ h_W)^*(\iota) & \text{if } c = 1; \end{cases}$$

and associate an integer $p(W^c_{l,k})$ to $W^c_{l,k}$ such that $p_1(W^c_{l,k}) = p(W^c_{l,k})\alpha$. It follows from the isomorphism $\mathcal{S}^{TOP}(W^0_{l,k}) \xrightarrow{\eta} [W^0_{l,k}, G/TOP]$ that

$$p(W^1_{l,k})\pi_W^*(\iota) = p_1(W^0_{l,k}) + f^* p_1(G/TOP)$$

where $f = d^{-1}(\pi_W^*(\iota))$ is the generator of $[W_{l,k}^0, G/TOP]$ corresponding to the generator $W_{l,k}^1$ and where $p_1(G/TOP)$ corresponds to the first Pontrjagin class in $H^4(BTOP)$ under the homomorphism $H^4(BTOP) \to H^4(G/TOP)$ induced by the natural map $G/TOP \to BTOP$. From [19, Proposition 13.4] we get

$$f^*p_1(G/TOP) = 24\pi_W^*(\iota).$$

This, together with the fact $p(W_{l,k}^0) = 2(k+2l)$ (see [14]) and the formula for $p(W_{l,k}^1)$ above, implies that $p(W_{l,k}^c) = 2k + 4l + 24c$. Consequently, as $M_{l,k}^c = \partial W_{l,k}^c$ we get iv).

Finally, we compute the invariant s_1 of $M_{l,k}^c$. The exact sequence

$$H^4(W_{l,k}^c, M_{l,k}^c) \xrightarrow{j^*} H^4(W_{l,k}^c) \to H^4(M_{l,k}^c) \to 0,$$

together with the isomorphisms $H^4(M_{l,k}^c) \cong \mathbb{Z}_k$ and $H^4(W_{l,k}^c, M_{l,k}^c) \cong \mathbb{Z}$ by the Poincaré duality, implies that we can take a generator β of $H^4(W_{l,k}^c, M_{l,k}^c)$ such that $j^*(\beta) = k\alpha$. Since $j^* : H^4(W_{l,k}^c, M_{l,k}^c; \mathbb{Q}) \to H^4(W_{l,k}^c; \mathbb{Q})$ is an isomorphism for $k \neq 0$ the invariant s_1 is defined for $M_{l,k}^c$. It follows from the Poincaré duality and the relation $j^*(\beta) = k\alpha$ that $\sigma(W_{l,k}^c) = \frac{|k|}{k}$. On the other hand, the formula for $p(W_{l,k}^c)$, together with the relation $j^*(\beta) = k\alpha$ and the isomorphism

$$H^4(W_{l,k}^c, M_{l,k}^c) \otimes H^4(W_{l,k}^c) \xrightarrow{\cup} H^8(W_{l,k}^c, M_{l,k}^c)$$

by the Poincaré duality, implies that

$$p_1^2(W_{l,k}^c) = \tfrac{4}{k}(k + 2l + 12c)^2.$$

In addition, the relation $w_2(W_{l,k}^1) = w_2(W_{l,k}^0) = 0$ indicates that we can take $z = 0$ in the formula of s_1. Substituting the values of $\sigma(W_{l,k}^c), z, p_1^2(W_{l,k}^c)$ in the formula for s_1, yields v).

Similarly, we get vi) shown by Crowley and Escher (see [3]).

This completes the computation of the invariant system $\{H, \Delta, q_1, b, s_1, \mu\}$ for the manifolds $M_{l,k}^c$.□

3.3 Circle bundles over $N \in \Theta$

It is shown in [6] that the invariant μ is additive with respect to connected sums and can be used to distinguish exotic $7-$spheres. According to $\mu(M_{1,1}^0) \equiv \frac{1}{28} \bmod \mathbb{Z}$, the group Γ_7 of exotic $7-$spheres is cyclic of order 28 and is generated by $M_{1,1}^0$. Let $\Sigma_r := rM_{1,1}^0 \in \Gamma_7$, $r \in \mathbb{Z}$.

Lemma 3.3. Let N_t be the total space of the oriented circle bundle over $N \in \Theta$ with Euler class the fixed generator $t \in H^2(N)$. Then there is a homeomorphism

$$N_t \cong M^c_{l,k}$$

with $(k,c,l) = (k(N), \frac{1+(-1)^{k(N)}}{2} \cdot \delta(N), \frac{p(N)+(3\varepsilon(N)-4)\cdot k(N)-(1+(-1)^{k(N)})\cdot 12\delta(N)}{4})$.

Furthermore, if $\frac{1+(-1)^{k(N)}}{2} \cdot \delta(N) = 0$ and $k(N) \neq 0$, then the manifold N_t has a smooth structure and one has a diffeomorphism

$$N_t \cong M^0_{l,k} \# \Sigma_r$$

with $(k,l,r) = (k(N), \frac{p(N)+(3\varepsilon(N)-4)\cdot k(N)}{4}, \frac{(1-\varepsilon(N))\cdot(p(N)-4k(N))}{24})$.

Proof. Since $\pi_i(PL/O) = 0$ for $i \leq 6$ the manifold N_t is smoothable if and only if $\Delta(N_t) = 0$ (see [13, p.33], [15] and [19, Theorem 5.4]). We divide the proof into two cases depending on whether N_t is smoothable.

Case 1. $\Delta(N_t) \equiv 0 \bmod 2$: In this case the system $\{H, q_1, b, s_1\}$ (resp. $\{H, q_1, b, \mu\}$) is a complete set of homeomorphism (resp. diffeomorphism) invariants for the manifolds $N_t, M^0_{l,k}$ (see [3] and [23]). The proof is completed by comparing the values of the invariants for $N_t, M^0_{l,k}$ obtained in Example 3.1 and 3.2, respectively.

Case 2. $\Delta(N_t) \equiv 1 \bmod 2$: We need to show that N_t is homeomorphic to $M^1_{l,k}$ with

$$(k,l) = (k(N), \frac{p(N)+(3\varepsilon(N)-4)\cdot k(N)-24}{4}).$$

It suffices to find a homotopy equivalence $q : N_t \to M^0_{l,k}$ such that $[N_t, q] = \left[M^1_{l,k}, h_M\right]$.

By Lemma 2.2, there exists a manifold $N' \in \Theta$ whose invariant system $(k(N'), p(N'), \varepsilon(N'), \delta(N'))$ is $(k(N), p(N)-24, \varepsilon(N), 0)$. Consider the map $\eta : \mathcal{S}^{TOP}(N') \to [N', G/TOP]$ in the surgery exact sequence of N'. By the argument at the end of the proof of [10, Theorem 1] we can find a manifold $N \in \Theta$ with a homotopy equivalence $h_N : N \to N'$ such that

i) the homotopy class $\eta([N, h_N])$ is trivial on the 2 skeleton of N';

ii) the primary obstruction to finding a null-homotopy of $\eta([N, h_N])$ is the fixed generator $x \in H^4(N')$.

Pulling back h_N by the bundle projection $\pi : N'_t \to N'$ induces a homotopy equivalence $h_t : N_t \to N'_t$. On the other hand, by the result of case 1 we get a homeomorphism $u : M^0_{l,k} \to N'_t$ such that $u^*(\pi^*(x)) = \pi^*_M(\iota)$. So it remains to show that $\left[N_t, u^{-1} \circ h_t\right] = \left[M^1_{l,k}, h_M\right]$.

Let $[N', G/TOP]_2$ be the subset of $[N', G/TOP]$ whose elements are trivial on the 2 skeleton of N' and consider the following two commutative diagrams:

$$\begin{array}{ccc}
\mathcal{S}^{TOP}(N^{'}) & \overset{\pi^*}{\to} & \mathcal{S}^{TOP}(N_t^{'}) \\
\downarrow \eta & & \cong\downarrow \eta \\
[N^{'}, G/TOP] & \overset{\pi^*}{\to} & [N_t^{'}, G/TOP]
\end{array}$$

$$\begin{array}{ccccc}
 & & \mathcal{S}^{TOP}(N_t^{'}) & \underset{\cong}{\overset{u^*}{\to}} & \mathcal{S}^{TOP}(M_{l,k}^0) \\
 & & \cong\downarrow \eta & & \cong\downarrow \eta \\
[N^{'}, G/TOP]_2 & \overset{\pi^*}{\to} & [N_t^{'}, G/TOP] & \underset{\cong}{\overset{u^*}{\to}} & [M_{l,k}^0, G/TOP] \\
\downarrow d & & \cong\downarrow d & & \cong\downarrow d \\
H^4(N^{'}) & \overset{\pi^*}{\to} & H^4(N_t^{'}) & \underset{\cong}{\overset{u^*}{\to}} & H^4(M_{l,k}^0)
\end{array}$$

where

i) $\mathcal{S}^{TOP}(N^{'}) \overset{\pi^*}{\to} \mathcal{S}^{TOP}(N_t^{'})$ maps $\left[N^{''}, h^{''}\right]$ to $\left[N_t^{''}, h_t^{''}\right]$ with $h_t^{''}$ a pull-back of $h^{''}$ by the bundle projection $\pi : N_t^{'} \to N^{'}$;

ii) $\mathcal{S}^{TOP}(N_t^{'}) \overset{u^*}{\to} \mathcal{S}^{TOP}(M_{l,k}^0)$ maps $\left[M^{'}, g^{'}\right]$ to $\left[M^{'}, u^{-1} \circ g^{'}\right]$;

iii) the maps d send a homotopy class to its primary obstruction to null-homotopy.

The diagrams above, together with the relations

$$\pi^*[N, h_N] = [N_t, h_t],\ u^*(\pi^*(x)) = \pi_M^*(\iota) \text{ and } d(\eta\,[N, h_N]) = x,$$

imply that $u^*[N_t, h_t] = \left[M_{l,k}^1, h_M\right]$, i.e. $\left[N_t, u^{-1} \circ h_t\right] = \left[M_{l,k}^1, h_M\right]$. This completes the proof of Case 2.□

4 The main results and their proofs

In the final section, we establish Theorem 1.3 stated in Section 1 and obtain its analogue in the smooth category.

Proof of Theorem 1.3. Let M be a 2–connected 7−manifold which admits a regular circle action. According to Lemma 2.1 and Lemma 2.2 M must be the total space of an oriented circle bundle over $N\#_r S^3 \times S^3$ with Euler class $\bar{t} \in H^2(N\#_r S^3 \times S^3) = \mathbb{Z}$ a generator, where $N \in \Theta$, $r \in \mathbb{N}$. We can regard $\bar{t} \in H^2(N\#_r S^3 \times S^3)$ corresponds to the fixed generator $t \in H^2(N) \cong \mathbb{Z}$ under the isomorphism $H^2(N) \to H^2(N\#_r S^3 \times S^3)$ induced by the map $N\#_r S^3 \times S^3 \to N$ collapsing $\#_r S^3 \times S^3$ to a point. By Lemma 2.2 and Lemma 3.3 it suffices for us to show that $M = N_t \#_{2r} S^3 \times S^4$.

The connected sum $N\#_r S^3 \times S^3$ is obtained by removing open 6−disks $\overset{\circ}{D}_1$ and $\overset{\circ}{D}_2$ from N and $\#_r S^3 \times S^3$ respectively, and pasting the resulting boundaries by a diffeomorphism $f : \partial D_2 \to \partial D_1$. Since the restriction of the bundle $N_t \to N$ on D_1 is trivial we have the decomposition

$$M = (N_t \backslash \overset{\circ}{D}_1 \times S^1) \cup_{f\times id} ((\#_r S^3 \times S^3 \backslash \overset{\circ}{D}_2) \times S^1) = N_t \# M_0.$$

where $id : S^1 \to S^1$ is the identity map, and where

$$M_0 = (S^7 \backslash \overset{\circ}{D}_1 \times S^1) \cup_{f\times id} ((\#_r S^3 \times S^3 \backslash \overset{\circ}{D}_2) \times S^1).$$

Since M_0 is easily seen to be the total space of the oriented circle bundle over $CP^3 \#_r S^3 \times S^3$ with Euler class a proper generator of $H^2(CP^3 \#_r S^3 \times S^3) = \mathbb{Z}$, a calculation similar to that in Example 3.1 shows that the invariant system $\{H, q_1, b, \mu\}$ for M_0 and $\#_{2r} S^3 \times S^4$ coincide. Consequently M_0 is diffeomorphic to $\#_{2r} S^3 \times S^4$. This shows that $M = N_t \#_{2r} S^3 \times S^4$.□

Let M be a smooth 2−connected 7−manifold which admits a smooth free circle action. Since such an action must be regular and the orbit manifold remains smooth (see [1, p.84] and [21, p.38, Proposition 5.2]), we still have the decomposition $M = N_t \#_{2r} S^3 \times S^4$ where $r \in \mathbb{N}, N \in \Theta$, and the manifold N_t must be smooth. From Lemma 2.2 and the second part of Lemma 3.3 we get

Theorem 4.1. Let M be a smooth 2−connected 7−manifold whose third homology group $H_3(M)$ is either of even rank, or has non trivial torsion subgroup. Then M admits a smooth free circle action if and only if M is diffeomorphic to

$$\#_{2r} S^3 \times S^4 \# M^0_{6(c+1)m,(c+1)k} \# \Sigma_{(1-c)m}$$

where $c \in \{0, 1\}$, $r \in \mathbb{N}$ and $m, k \in \mathbb{Z}$.□

A classical topic in topology is to decide which homotopy spheres admit smooth free circle action, see [8] [12] [17] [18] for instance. Combining Theorem 4.1 with the computation in Example 3.2 we get the following result shown by [17] .

Corollary 4.2. Among the 28 homotopy 7–spheres $\{\Sigma_r \mid 0 \le r \le 27\}$ only the following ones admit smooth free circle actions

$$\{\Sigma_r \mid r = 0, 4, 6, 8, 10, 14, 18, 20, 22, 24\}.\square$$

In the smooth category and in terms of Theorem 4.1, the problem remains open for the 2−connected 7−manifolds whose third homology group $H_3(M)$ is free and is of odd rank. The reason is that there is no complete diffeomorphism invariant system available for the 2−connected 7−manifolds M with $H_3(M) = \mathbb{Z}$.

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